OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS 4726

Further Pure Mathematics 2

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Monday 20 June 2011
Morning

Duration: 1 hour 30 minutes

* 4 7 2 6 *

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a scientific or graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 Express $\dfrac{2x+3}{(x+3)(x^2+9)}$ in partial fractions. **[5]**

2 A curve has equation $y = \dfrac{x^2 - 6x - 5}{x-2}$.

(i) Find the equations of the asymptotes. **[3]**

(ii) Show that y can take all real values. **[4]**

3 It is given that $\mathrm{F}(x) = 2 + \ln x$. The iteration $x_{n+1} = \mathrm{F}(x_n)$ is to be used to find a root, α, of the equation $x = 2 + \ln x$.

(i) Taking $x_1 = 3.1$, find x_2 and x_3, giving your answers correct to 5 decimal places. **[2]**

(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 3.146\,19$, correct to 5 decimal places, use the values of e_2 and e_3 to make an estimate of $\mathrm{F}'(\alpha)$ correct to 3 decimal places. State the true value of $\mathrm{F}'(\alpha)$ correct to 4 decimal places. **[3]**

(iii) Illustrate the iteration by drawing a sketch of $y = x$ and $y = \mathrm{F}(x)$, showing how the values of x_n approach α. State whether the convergence is of the 'staircase' or 'cobweb' type. **[3]**

4 A curve C has the cartesian equation $x^3 + y^3 = axy$, where $x \geqslant 0$, $y \geqslant 0$ and $a > 0$.

(i) Express the polar equation of C in the form $r = \mathrm{f}(\theta)$ and state the limits between which θ lies. **[3]**

The line $\theta = \alpha$ is a line of symmetry of C.

(ii) Find and simplify an expression for $\mathrm{f}\left(\frac{1}{2}\pi - \theta\right)$ and hence explain why $\alpha = \frac{1}{4}\pi$. **[3]**

(iii) Find the value of r when $\theta = \frac{1}{4}\pi$. **[1]**

(iv) Sketch the curve C. **[2]**

5 **(i)** Prove that, if $y = \sin^{-1} x$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{1-x^2}}$. **[3]**

(ii) Find the Maclaurin series for $\sin^{-1} x$, up to and including the term in x^3. **[5]**

(iii) Use the result of part **(ii)** and the Maclaurin series for $\ln(1+x)$ to find the Maclaurin series for $(\sin^{-1} x)\ln(1+x)$, up to and including the term in x^4. **[4]**

6 It is given that $I_n = \displaystyle\int_0^1 x^n (1-x)^{\frac{3}{2}}\,\mathrm{d}x$, for $n \geqslant 0$.

(i) Show that $I_n = \dfrac{2n}{2n+5} I_{n-1}$, for $n \geqslant 1$. **[6]**

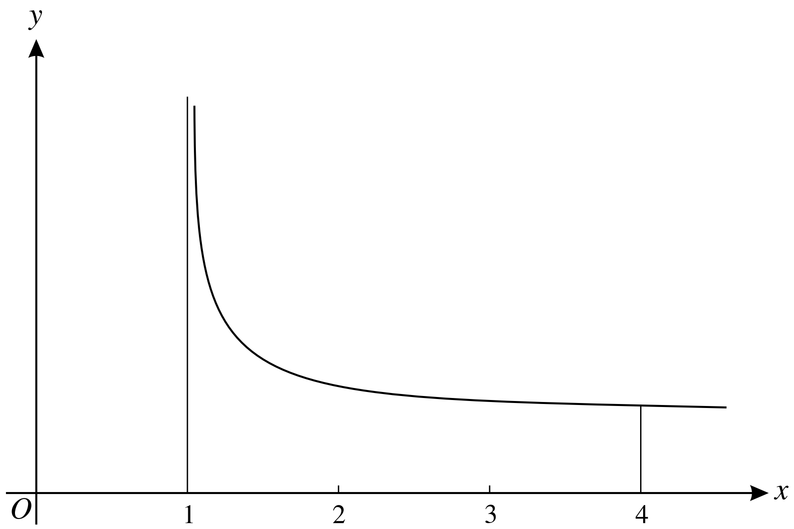
(ii) Hence find the exact value of I_3. **[4]**

7 **(i)** Sketch the graph of $y = \tanh x$ and state the value of the gradient when $x = 0$. On the same axes, sketch the graph of $y = \tanh^{-1} x$. Label each curve and give the equations of the asymptotes. **[4]**

(ii) Find $\int_0^k \tanh x \, dx$, where $k > 0$. **[2]**

(iii) Deduce, or show otherwise, that $\int_0^{\tanh k} \tanh^{-1} x \, dx = k \tanh k - \ln(\cosh k)$. **[4]**

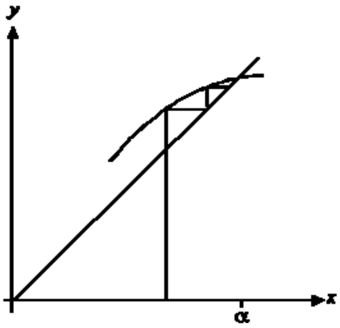
8 **(i)** Use the substitution $x = \cosh^2 u$ to find $\int \sqrt{\frac{x}{x-1}} \, dx$, giving your answer in the form $f(x) + \ln\big(g(x)\big)$. **[7]**

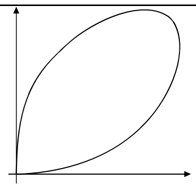


(ii) Hence calculate the exact area of the region between the curve $y = \sqrt{\frac{x}{x-1}}$, the x-axis and the lines $x = 1$ and $x = 4$ (see diagram). **[1]**

(iii) What can you say about the volume of the solid of revolution obtained when the region defined in part **(ii)** is rotated completely about the x-axis? Justify your answer. **[3]**

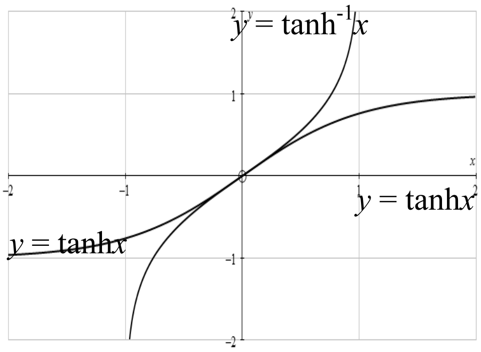
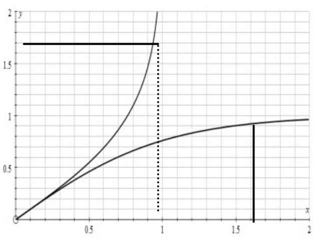
1	$\dfrac{2x+3}{(x+3)(x^2+9)} \equiv \dfrac{A}{x+3}+\dfrac{Bx+C}{x^2+9}$	**B1**	For correct form seen anywhere with letters or values
	$A=-\dfrac{1}{6}$	**B1**	For correct A (cover up or otherwise)
	$2x+3 \equiv A(x^2+9)+(Bx+C)(x+3)$	**M1**	For equating coefficients at least once.(or substituting values) into correct identity.
	$B=\dfrac{1}{6},\quad C=\dfrac{3}{2}$	**A1**	For correct B and C
	$\Rightarrow \dfrac{-1}{6(x+3)}+\dfrac{x+9}{6(x^2+9)}$	**A1**	For correct final statement cao, oe
		5	
2(i)	Asymptote $x=2$	**B1**	For correct equation
	$y=x-4-\dfrac{13}{x-2}$	**M1**	For dividing out (remainder not required)
	$\Rightarrow$ asymptote $y=x-4$	**A1** **3**	For correct equation of asymptote (ignore any extras)
(ii)	METHOD 1		**N.B. answer given**
	$x^2-(y+6)x+(2y-5)=0$	**M1**	For forming quadratic in x
	$b^2-4ac(\geq 0) \Rightarrow (y+6)^2-4(2y-5)(\geq 0)$	**M1**	For considering discriminant
	$\Rightarrow y^2+4y+56(\geq 0)$	**A1**	For correct simplified expression in y **soi**
	$\Rightarrow (y+2)^2+52 \geq 0$: this is true $\forall y$ So y takes all values	**A1**	For completing square (or equivalent) and correct conclusion **www**
	METHOD 2		
	Obtain $\dfrac{dy}{dx}=\dfrac{x^2-4x+17}{(x-2)^2}$ OR $1+\dfrac{13}{(x-2)^2}$	**M1**	For finding $\dfrac{dy}{dx}$ either by direct differentiation or dividing out first
		A1	For correct expression **oe.**
	$\Rightarrow \dfrac{dy}{dx} \geq 1\ \forall x$,	**M1**	For drawing a conclusion
	so y takes all values.	**A1**	For correct conclusion **www**
		4	
	Alternate scheme: Sketching graph		
	Graph correct approaching asymptotes from both side	**B1**	A graph with no explanation can only score 2
	Graph completely correct	**B1**	
	Explanation about no turning values	**B1**	
	Correct conclusion	**B1**	

3(i)	$x_1 = 3.1 \Rightarrow x_2 = 3.13140$, $x_3 = 3.14148$	**B1** **B1** **2**	For correct x_2 For correct x_3
(ii)	$\mathrm{F}'(\alpha) \approx \frac{e_3}{e_2} = \frac{0.00471}{0.01479} = 0.318\ (0.31846)$ $\mathrm{F}'(\alpha) = \frac{1}{\alpha} = 0.3178\ (0.31784)$	**M1** **A1** **B1** **3**	For dividing e_3 by e_2 For estimate of $\mathrm{F}'(\alpha)$ For true $\mathrm{F}'(\alpha)$ obtained from $\frac{\mathrm{d}}{\mathrm{d}x}(2+\ln x)$ **TMDP anywhere in (i) (ii) deduct 1 once (but answers must round to given values or A0)**
(iii)	 Staircase	**B1** **B1** **B1** **3**	For $y = x$ and $y = \mathrm{F}(x)$ drawn, crossing as shown For lines drawn to illustrate iteration (Min 2 horizontal and 2 vertical seen) For stating "staircase"

4(i)	$x = r\cos\theta,\ y = r\sin\theta$ $\Rightarrow r = \dfrac{a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$ for $0 \le \theta \le \dfrac{1}{2}\pi$	**M1** **A1** **A1** **3**	For substituting for x and y For correct equation **oe** (Must be $r = \ldots\ldots$) For correct limits for θ (Condone <)
(ii)	$\mathrm{f}\left(\frac{1}{2}\pi - \theta\right) = \dfrac{a\cos\left(\frac{1}{2}\pi - \theta\right)\sin\left(\frac{1}{2}\pi - \theta\right)}{\cos^3\left(\frac{1}{2}\pi - \theta\right) + \sin^3\left(\frac{1}{2}\pi - \theta\right)}$ $= \dfrac{a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$ $\mathrm{f}(\theta) = \mathrm{f}\left(\frac{1}{2}\pi - \theta\right) \Rightarrow \alpha = \frac{1}{4}\pi$	**M1** **A1** **A1** **3**	**N.B. answer given** For replacing θ by $\left(\frac{1}{2}\pi - \theta\right)$ in their $\mathrm{f}(\theta)$ For correct simplified form. (Must be convincing) For correct reason for $\alpha = \frac{1}{4}\pi$
(iii)	$r = \dfrac{a.\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3} = \frac{1}{2}\sqrt{2}\,a$	**B1** **1**	For correct value of r.**oe**
(iv)		**B1** **B1** **2**	Closed curve in 1st quadrant only, symmetrical about $\theta = \frac{1}{4}\pi$ Diagram showing $\theta = 0, \frac{1}{2}\pi$ tangential at O

5(i)	$x=\sin y \Rightarrow \frac{dx}{dy}=\cos y$	**M1**	For implicit diffn to $\frac{dy}{dx}=\pm\frac{1}{\cos y}$ **oe**
	$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$	**A1**	For using $\sin^2 y+\cos^2 y=1$ to obtain **N.B. Answer given**
	$+\sqrt{\ }$ taken since $\sin^{-1}x$ has positive gradient	**B1** 3	For justifying + sign
(ii)	$f(0)=0,\ f'(0)=1$	**B1**	For correct values
	$f''(x)=\frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}$	**M1**	Use of chain rule to differentiate $f'(x)$
	$f'''(x)=\frac{\left(1-x^2\right)^{\frac{3}{2}}+3x^2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^3}$	**M1**	Use of quotient or product rule to differentiate f'' (0).
	$\Rightarrow f''(0)=0,\quad f'''(0)=1$	**A1**	For correct values **www, soi**
	$\Rightarrow \sin^{-1}x=x+\frac{1}{6}x^3$	**A1** 5	For correct series (allow 3!) **www**
	Alternative Method: f(0) = 0, f'(0) = 1	**B1**	For correct values
	$f'(x)=\frac{1}{\sqrt{1-x^2}}=\left(1-x^2\right)^{-\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{3}{8}x^4+...$	**M1**	Correct use of binomial
	$f''(x)=x+\frac{3}{2}x^3+...$	**M1**	Differentiate twice
	$f'''(x)=1+\frac{9}{2}x^2+...$		
	$\Rightarrow f'(0)=1, f''(0)=0, f'''(0)=1$	**A1**	Correct values
	$\Rightarrow \sin^{-1}x=x+\frac{1}{6}x^3$	**A1**	Correct series
(iii)	$\left(\sin^{-1}x\right)\ln(1+x)$ $=\left(x+\frac{1}{6}x^3\right)\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3\right)$	**B1ft**	For terms in both series to at least x^3 f.t. from their **(ii)** multiplied together
	$=x^2-\frac{1}{2}x^3+\frac{1}{2}x^4$	**M1**	For multiplying terms to at least x^3
		A1	For correct series up to x^3 **www**
		A1 4	For correct term in x^4 **www**

6(i)	$I_n = \int_0^1 x^n (1-x)^{\frac{3}{2}}\ dx$	**M1**	For integrating by parts (correct way round)
	$= \left[-\frac{2}{5}x^n(1-x)^{\frac{5}{2}}\right]_0^1 + \frac{2}{5}n\int_0^1 x^{n-1}(1-x)^{\frac{5}{2}}\ dx$	**A1**	For correct first stage
	$\Rightarrow I_n = \frac{2}{5}n\int_0^1 x^{n-1}(1-x)^{\frac{5}{2}}\ dx$	**A1**	
	$\Rightarrow I_n = \frac{2}{5}n\int_0^1 x^{n-1}(1-x)(1-x)^{\frac{3}{2}}\ dx$	**M1**	For splitting $(1-x)^{5/2}$ suitably
	$\Rightarrow I_n = \frac{2}{5}nI_{n-1} - \frac{2}{5}nI_n$	**A1**	For obtaining correct relation between I_n and I_{n-1}
	$\Rightarrow I_n = \frac{2n}{2n+5}I_{n-1}$	**A1** **6**	For correct result (**N.B. answer given**)
(ii)	$I_0 = \left[-\frac{2}{5}(1-x)^{\frac{5}{2}}\right]_0^1 = \frac{2}{5}$	**M1**	For evaluating I_0 [*OR* I_1 by parts]
		M1	For using recurrence relation 3 [*OR* 2] times (may be combined together)
	$I_3 = \frac{6}{11}I_2 = \frac{6}{11}\times\frac{4}{9}I_1 = \frac{6}{11}\times\frac{4}{9}\times\frac{2}{7}I_0$	**A1**	For 3 [*OR* 2] correct fractions
	$I_3 = \frac{32}{1155}$	**A1** **4**	For correct exact result

7(i)		B1 B1 B1 B1 4	Both curves of the correct shape (ignore overlaps) and labelled gradient = 1 at $x = 0$ stated For asymptotes $y = \pm 1$ and $x = \pm 1$ (or on sketch) Sketch all correct
(ii)	$\int_0^k \tanh x \, dx = [\ln(\cosh x)]_0^k = \ln(\cosh k)$	M1 A1 2	For substituting limits into $\ln \cosh x$ For correct answer
(iii)	 Areas shown are equal: $x = \tanh k$ $\Rightarrow y = k$ $\Rightarrow \int_0^{\tanh k} \tanh^{-1} x \, dx$ = rectangle ($k \times \tanh k$)– **(ii)** $= k \tanh k - \ln(\cosh k)$	M1 A1 M1 A1 4	For consideration of areas For sufficient justification For subtraction from rectangle For correct answer **N.B. answer given** **Alternative:** Otherwise by parts, as $1 \times \tanh^{-1} x$ *OR* $1 \times \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

PTO for alternative schemes

7(iii)	Alternative method 1 By parts: $I = \int_0^{\tanh k} \tanh^{-1} x \, dx$ $u = \tanh^{-1} x \quad dv = dx$ $du = \frac{1}{1-x^2} dx \quad v = x$ $\Rightarrow I = \left[x \tanh^{-1} x\right]_0^{\tanh k} - \int_0^{\tanh k} \frac{x}{1-x^2} dx$ $= k \tanh k + \frac{1}{2}\left[\ln(1-x^2)\right]_0^{\tanh k}$ $= k \tanh k + \frac{1}{2}\ln(1-\tanh^2 k)$ $= k \tanh k + \frac{1}{2}\ln(\text{sech}^2 k)$ $= k \tanh k + \ln(\text{sech} k)$	**M1** **A1** **M1** **A1**	For integrating by parts (correct way round) For getting this far Dealing with the resulting integral
	Alternative method 2 By substitution Let $y = \tanh^{-1} x \Rightarrow x = \tanh y$ $\Rightarrow dx = \text{sech}^2 y \, dy$ When $x = 0,\ y = 0$ When $x = \tanh k,\ y = k$ $\Rightarrow I = \int_0^{\tanh k} \tanh^{-1} x \, dx = \int_0^k y \text{sech}^2 y \, dy$ $u = y \quad dv = \text{sech}^2 y \, dy$ $du = dy \quad v = \tanh y$ $\Rightarrow I = \left[y \tanh y\right]_0^k - \int_0^k \tanh y \, dy$ $= k \tanh k - \ln \cosh k$	**M1** **A1** **M1** **A1**	For substitution to obtain equivalent integral Correct so far For integration by parts (correct way round) Final answer

8(i)	$x = \cosh^2 u \Rightarrow \mathrm{d}u = 2\cosh u \sinh u\,\mathrm{d}u$	**B1**	For correct result
	$\int \sqrt{\frac{x}{x-1}}\,\mathrm{d}x = \int \frac{\cosh u}{\sinh u} 2\cosh u \sinh u\,\mathrm{d}u$	**M1**	For substituting throughout for x
	$= \int 2\cosh^2 u\,\mathrm{d}u$	**A1**	For correct simplified u integral
	$= \int (\cosh 2u + 1)\,\mathrm{d}u = \sinh u \cosh u + u$	**M1**	For attempt to integrate $\cosh^2 u$
		A1	For correct integration
	$= x^{\frac{1}{2}}(x-1)^{\frac{1}{2}} + \ln\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)\ (+c)$	**M1**	For substituting for u
		A1 **7**	For correct result **oe** as $\mathrm{f}(x) + \ln(\mathrm{g}(x))$
(ii)	$2\sqrt{3} + \ln(2 + \sqrt{3})$	**B1** **1**	
(iii)	$V = (\pi)\int_1^4 \frac{x}{x-1}\,\mathrm{d}x = (\pi)\left[x + \ln(x-1)\right]_1^4$	**M1**	For attempt to find $\int \frac{x}{x-1}\,\mathrm{d}x$
		A1	For correct integration (ignore π)
	$V \to \infty$	**B1** **3**	For statement that volume is infinite (independent of M mark)